J. Appl. Maths Mechs Vol. 56, No. 2, pp. 293–297, 1992
Printed in Great Britain.

0021–8928/92 \$15.00+.00

THE LOWER BOUND OF THE EIGENFREQUENCIES OF PLANE OSCILLATIONS OF A FLUID IN A CHANNEL†

N. G. KUZNETSOV

St Petersburg

(*Received* 25 *April* 1991)

A method is proposed for estimating the lower bound of the eigenfrequencies of free plane oscillations of an ideal fluid in a channel of constant cross-section. The method is based on the identity [1] which is used to prove the theorem of uniqueness in the problem of radiation and scattering of the waves by a body completely immersed in the fluid. Unlike in the estimates obtained earlier [2, 3],‡ here the bottom of the channel is not explicitly specified. It is shown that in a number of cases the estimate can be substantially improved when it is used together with the principle of monotony [4, 5]. The method developed here can also be used for a three-dimensional problem. Examples are solved and the results compared with the known exact and approximate values [5, 6] and with the estimates obtained by others [7].‡ A lower bound of a different type was obtained in [8]. A bibliography of the problems of eigenfrequencies of the oscillations of an ideal fluid can be found in [5, 9].

1. FORMULATION OF THE PROBLEM

LET A channel of cross-section W be filled with an ideal incompressible fluid equilibrium. The simple-connected region $W \subset R_-^2 = \{(x, y)\colon\ y<0\}$ has a piecewise-smooth boundary without the cusps, and $\partial W = F \cup B$, $F \cap B = \varnothing$. Here $F = \{x \in (-a, +a);\ y = 0\}$ is the free surface of the fluid ($a>0$), B is the bottom of the channel. The bottom represents a curve which lies within R_-^2 with the exception of its ends $(\pm a, 0)$ which are the angle points of ∂W.

We use the mixed Steklov problem [4–6, 9] to describe the plane free oscillations of the fluid in the channel, harmonic in time:

$$\nabla^2 u = 0 \text{ in } W,\quad u_y = \nu u \text{ on } F,\quad \partial u/\partial n = 0 \text{ on } B \cap R_-^2 \tag{1.1}$$

This is a spectral problem, in which we require to find the eigenvalues of the parameter ν and the corresponding real eigenfunctions belonging to the Sobolev space $H^1(W)$ and satisfying the condition $\int_F u\,dx = 0$. Let $\mathbf{n}$ be the inner normal to ∂W, the quantity νg (g is the acceleration due to gravity) be equal to the square of the frequency of free oscillations and u be the velocity potential of the oscillations apart from a multiplier harmonic in time.

We know [4, 5] that problem (1.1) has a discrete spectrum $0<\nu_1<\nu_2\leqslant\nu_3\leqslant\ldots\leqslant\nu_n<\ldots$

2. AUXILIARY IDENTITY

The identity [1]

$$\begin{gathered} 2\nabla\cdot[(\mathbf{V}\cdot\nabla u+Hu)\nabla u]=2(\mathbf{V}\cdot\nabla u+Hu)\nabla^2 u- \\ -(Q\nabla u)\cdot\nabla u+\nabla\cdot(|\nabla u|^2\mathbf{V}+u^2\nabla H)-u^2\nabla^2 H \end{gathered} \tag{2.1}$$

†*Prikl. Mat. Mekh.* Vol. 56, No. 2, pp. 342–346, 1992.

‡See also: Fedorov A. L., Geometrical estimates of capacity-type functionals and eigenvalues for regions of complex form. Candidate dissertation, Kiev, 1973.

can be verified directly. Here $\mathbf{V} = (V_1, V_2)$ is the vector field on $\bar{W}$, whose components are real and satisfy uniformly the Lipshitz condition on $\bar{W}$; H is a real function on $\bar{W}$ with first-order derivatives satisfying uniformly the Lipshitz condition. The matrix Q has the elements $Q_{ij} = (\nabla \cdot \mathbf{V} - 2H)\delta_{ij} - (\partial V_i/\partial x_j + \partial V_j/\partial X_i)$, δ_{ij} is the Kronecker delta and $x_1 = x$, $x_2 = y$.

Let u be the eigenfunction of problem (1.1) corresponding to the smallest eigenvalue ν^1. Let us integrate the identity (2.1) over the region W, and use the Laplace equation and Gauss' theorem. As a result we obtain

$$\int_W [(Q\nabla u)\cdot\nabla u + u^2\nabla^2 H]\,dx\,dy = \\ = \int_{\partial W} [2(\mathbf{V}\cdot\nabla u + Hu)\partial u/\partial n - \mathbf{V}\cdot\mathbf{n}|\nabla u|^2 - u^2\partial H/\partial n]\,ds \tag{2.2}$$

Let us consider the last integral on the subset $F \subset \partial W$. If we take into account the boundary condition on F in (1.1), then the integral will be equal to

$$\int_F [V_2 u_x^2 - (\nu_1^2 V_2 + 2\nu_1 H - H_y)u^2]\,dx - 2\nu_1 \int_F V_1 u_x u\,dx$$

Integrating the last integral by parts, substituting the result into Eq. (2.2) and taking into account the boundary condition on B in (1.1), we arrive at the following identity:

$$\int_W [(Q\nabla u)\cdot\nabla u + u^2\nabla^2 H]\,dx\,dy + \\ + \int_B (\mathbf{V}\cdot\mathbf{n}|\nabla u|^2 + u^2\partial H/\partial n)\,ds + \\ + \int_F [\nu_1^2 V_2 + \nu_1(2H - \partial V_1/\partial x) - H_y]u^2\,dx - \\ - \int_F V_2 u_x^2\,dx = -\nu_1[V_1(x,0)u^2(x,0)]_{x=-a}^{x=+0} \tag{2.3}$$

3. DERIVATION OF THE ESTIMATE

Let us put $H = -\frac{1}{2}$ in (2.3) and require that the field $\mathbf{V}$ satisfy the condition

$$V_1 = -x, \quad V_2 = 0 \text{ on } F \tag{3.1}$$

Then the identity (2.3) will take the form

$$\int_W (Q\nabla u)\cdot\nabla u\,dx\,dy + \int_B |\nabla u|^2 \mathbf{V}\cdot\mathbf{n}\,ds = \nu_1 a \sum_{\pm} u^2(\pm a, 0)$$

Further, let the matrix Q be non-negative in the region W and

$$\inf\{\mathbf{V}\cdot\mathbf{n}:\ (x, y)\in B\} = m > 0 \tag{3.2}$$

Here from the last relation it follows that

$$m\int_B (\partial u/\partial s)^2\,ds \leqslant \nu_1 a \sum_{\pm} u^2(\pm a, 0) \tag{3.3}$$

Here the boundary condition on B in (1.1) was also taken into account. It was shown in [10] that the function u has a single nodal line whose ends lie on the free surface F and on the bottom B. The latter end divides the bottom into two parts B_+ and B_-, and the point $(\pm a, 0)$ serves as the second end of the curve $B_\pm$. According to the Newton–Leibnitz formula we have

$$|u(\pm a, 0)| = \left| \int_{B_\pm} (\partial u/\partial s)\,ds \right|$$

From this we obtain, using the Cauchy inequality,

$$u^2(\pm a, 0) \leqslant |B_\pm| \int\limits_{B_\pm} (\partial u/\partial s)^2\, ds \tag{3.4}$$

and therefore

$$\sum_{\pm} u^2(\pm a, 0) \leqslant |B| \int\limits_{B} (\partial u/\partial s)^2\, ds$$

Here $|B|$ is the length of the curve B (we have similarly $|B_\pm|$).
Combining the last inequality with (3.3), we arrive at the following result.

Theorem. Let the vector field $\mathbf{V}$ be such that the matrix Q is non-negative in the region W when $H = -1/2$, and conditions (3.1) and (3.2) hold. Then the following inequality holds:

$$m|B|^{-1} \leqslant \nu_1 a \tag{3.5}$$

Corollary. For the regions symmetric about the ordinate the estimate (3.5) can be replaced by

$$2m|B|^{-1} \leqslant \nu_1 a \tag{3.6}$$

Proof. By virtue of the symmetry, the end of the nodal line of the function u lying at the bottom divides B in half, i.e. $|B_+| = |B_-| = |B|/2$ in (3.4). Then from (3.3) and (3.4) we obtain (3.6).

Notes. (1) The corollary shows that in some cases the lower bound of the first eigenvalue ν_1 in the non-symmetric region W will be obtained more easily, if, instead of using inequality (3.5), we carry out the following scheme. We construct an auxiliary symmetric region W' so that $W' \subset W$ and the free surfaces of these two regions coincide. Further, using the inequality (3.6) we obtain the lower bound of the first eigenvalue ν_1' in the region W'. Since $\nu_1' \leqslant \nu_1$ by virtue of the principle of monotony (see e.g. [4, 5]), it follows that the resulting estimate will also hold for the eigenvalue ν_1. It may even be found better than the estimate obtained using the inequality (3.5) directly (see e.g. [4, chapter 4]).

(2) The method of obtaining the lower bound for the eigenvalues given here can also be used in the three-dimensional case. The corresponding vector fields can be obtained by rotating the plane fields about the ordinate (see [1]).

4. EXAMPLES

Let us write

$$\mathbf{V} = (-x, -ky), \quad 0 \leqslant k \leqslant 2 \tag{4.1}$$

Then, when $H = -1/2$, the matrix

$$Q = \begin{Vmatrix} 2-k & 0 \\ 0 & k \end{Vmatrix}$$

will be non-negative. Condition (3.1) will also hold. Let us consider several specific regions satisfying the inequality (3.2) with the field (4.1), for any $k \in [0, 2]$. The estimates obtained for these regions from formulas (3.5) and (3.6) are now compared with the known exact results and the esimates obtained by others. We shall make the comparisons using the dimensionless quantity $\nu_1 a$.

Example 1. For a rectangular region W of width $2a$ and depth d, we put $k = 2$ in formula (4.1). Then

$$\min\{\mathbf{V}\cdot\mathbf{n}\colon (x, y) \in \bar{B}\} = \min\{a, 2d\}$$

and from inequality (3.6) we find that

$$\nu_1 a \geqslant \min\{a, 2d\}/(a+d) \tag{4.2}$$

Using the method of separation of variables we find that for a rectangle

TABLE 1

Type of channel cross-section	Exact value	Estimate obtained in this paper	Estimate obtained in [7]
Rectangle $d/a = 1/4$	0.58	$2/5$	—
Rectangle $d/a = 1/2$	1.04	$2/3$	0.09
Rectangle $d/a = 1$	1.44	$1/2$	—
Rectangle $d/a = 2$	1.57	$1/3$	0.14
Semicircle	1.36 [5]	$2/\pi$	0.08
Right-angle isosceles triangle $\alpha = \pi/4$	1 [6]	$1/2$	0.59
Isosceles triangle $\alpha = \pi/6$	—	$3/4$	0.33

$$\nu_1 a = \frac{\pi}{2} \operatorname{th} \frac{\pi d}{2a} \tag{4.3}$$

Thus, estimate (4.2) works quite well for small values of d. For large d formula (4.2) gives an estimate for $\nu_1 a$ which is much too low. Table 1 gives numerical results for some specific values on the ratio d/a enabling a comparison to be made of the estimate (4.2) with the exact value given by (4.3) and the estimate given in [7].

Example 2. For the region W which is a semicircle of radius a, we put $k = 1$ in (4.1). Then $\mathbf{V}\cdot\mathbf{n} = a$ on B and from the inequality (3.6) we find that $\nu_1 a \geq 2/\pi$ (compare with the exact value given in Table 1).

Example 3. Let the region W be an isosceles triangle with base of length $2a$, and let it serve as the free surface. We denote the angle at the base by α. If $k = 1$ in (4.1), then we find that $\mathbf{V}\cdot\mathbf{n} = a\sin\alpha$ on B. Then the inequality (3.6) will take the form

$$\nu_1 a \geq {}^1/_2 \sin 2\alpha \tag{4.4}$$

(compare with the exact value for $\alpha = \pi/4$ in Table 1).

The following estimate was obtained for the isosceles triangles by Fedorov (see the earlier footnote)

$$\nu_1 a \geq \frac{\pi}{2} \frac{(1+4\operatorname{tg}^2\alpha)^{1/2}-1}{(1+4\operatorname{tg}^2\alpha)^{1/2}+1}$$

which is worse than (4.4) for small values of α, and better for values of α approaching the value of $\pi/2$ (see Table 1).

Example 4. Let the region B be a right isosceles triangle whose leg represents the free surface. If $k = 1$ in (4.1), then $\mathbf{V}\cdot\mathbf{n} = a$ on the vertical part of the bottom, and $\mathbf{V}\cdot\mathbf{n} = a/\sqrt{2}$ on the sloping part of the bottom. Then equality (3.5) will take the form

$$\nu_1 a \geq (4+2\sqrt{2})^{-1} \tag{4.5}$$

Let us consider the right isosceles triangle W' for which the same free surface acts as the base. According to inequality (4.4) and note (1), we have $\nu_1 a \geq \nu_1' a \geq 1/2$. The last estimate is significantly better than (4.5).

Example 5. Using the same arguments we can obtain, by elementary methods, quite good estimates for sufficiently complex regions. Let us consider the region W shown in Fig. 1, and inscribe in it an isosceles

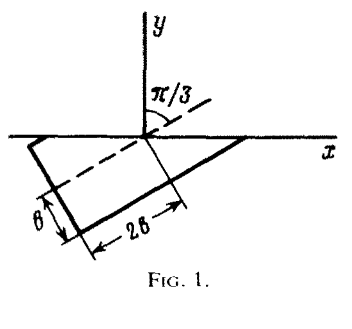

FIG. 1.

triangle with base angle equal to $\pi/6$, for which we shall use inequality (4.4). According to the principle of monotony the estimate $\nu_1 a \geqslant \sqrt{3}/4$ holds for the region W. The computed value of $\nu_1 a$ for the region shown in the figure is equal to 0.86.

Comparing the first two columns in Table 1 we see that the quantities obtained for the lower bound of $\nu_1 a$ vary within the limits of 21–69% of the exact value. The mean value in the second column is equal to 43% of the value in the first column.

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Translated by L.K.